

INVOLUTIONS ON ALGEBRAIC SURFACES AND ZERO CYCLES

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ABSTRACT. In this note we are going to consider a smooth projective surface equipped with an involution and study the action of the involution at the level of Chow group of zero cycles.

1. INTRODUCTION

In this note we want to consider the generalised Bloch conjecture which says the following. Let Γ be a correspondence of codimension 2 on $S \times T$ where S, T are smooth projective surfaces over the field of complex numbers. Suppose that Γ^* vanishes on $H^0(T, \Omega_T^2)$ then the homomorphism Γ_* from $CH_0(S)$ to $CH_0(T)$ vanishes on the kernel of the albanese map $alb_S : CH_0(S) \rightarrow Alb(S)$.

In the paper by [Voi] the conjecture was proved for a symplectic involution on a K3 surface, that is an automorphism i of the given K3 surface, such that i^* acts as identity on globally holomorphic 2-forms, then i_* acts as identity on CH_0 of the K3 surface. Also the similar question was considered in [G] for intersection of quadrics and cubics in \mathbb{P}^4 which are examples of K3 surfaces. Also in [HK] the question was considered and proved for certain examples of K3 surfaces equipped with a symplectomorphism.

In this note we consider the example of a Hilbert modular surface of general type, which admit a degree two canonical morphism onto a cubic in \mathbb{P}^3 . So the Hilbert modular surface itself is equipped with an involution. We check that this involution acts as -1 on the albanese kernel.

Our method is the same as in the proof of Bloch's conjecture for surfaces not of general type with $p_g = 0$ as in [BKL]. We consider the Hilbert modular surface and produce a dominant map from it to an elliptic surface fibered over the projective line. Then consider the Jacobian fibration associated to this elliptic surface and prove that it is rational resulting the -1 action of the involution on the albanese kernel of the Hilbert modular surface itself.

In the next section we try to understand the action of the involution on the Chow group of the surface, by reducing the problem to a very general hyperplane section of the surface itself. Precisely speaking let τ be the involution on the surface F . Let j_t denote the embedding of a very general hyperplane section C_t into F and $j_{\tau(t)}$ the embedding of $\tau(C_t)$ into F . Then we consider to study the kernel of the bi-Gysin homomorphism $j_{t*} + j_{\tau(t)*}$ from $J(C_t) \times J(\tau(C_t))$ to $A^2(F)$ (A^i is the group of algebraically trivial codimension i algebraic cycles modulo rational equivalence). By monodromy argument we prove that the kernel is either countable or τ acts as -1 on the image of $J(C_t)$ under j_{t*} into $A^2(F)$. Similar techniques will show that the kernel of $j_{t*} - j_{\tau(t)*}$ is either countable or τ acts as identity on the image of $J(C_t)$ under j_{t*} . Then combining these two facts we show that for a surface with $p_g > 0$ and $q = 0$, we have three possibilities, that either both the kernels $j_{t*} + j_{\tau(t)*}$, $j_{t*} - j_{\tau(t)*}$ are countable or τ acts as -1 on the image of $J(C_t)$ under j_{t*} or as identity on the image of $J(C_t)$ under j_{t*} . So if we can exclude the first possibility then we know the action of τ on $A^2(F)$, by studying the action of τ on the Jacobian of a very general hyperplane section.

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We assume that the ground field is algebraically closed and of characteristic zero.

2. INVOLUTION ON AN EXAMPLE OF HILBERT MODULAR SURFACE

Let K be a real quadratic field of discriminant D and let \mathfrak{o} be its ring of integers. The group $G = \mathbf{SL}_2(\mathfrak{o})/\{-\mathbf{1}, \mathbf{1}\}$ acts on $\mathcal{H} \times \mathcal{H}$, where \mathcal{H} is the upper half plane. The quotient space \mathcal{H}^2/G can be made compact with finitely many cusp singularities. This will be a normal complex algebraic surface, which has finitely many singular points. Resolving the singularities one obtains a non-singular algebraic surface denoted by $S(D)$, corresponding to the discriminant D . So for any discriminant D the surface $S(D)$ is defined and it is known as the Hilbert modular surface with discriminant D . For a classification of Hilbert modular surfaces please see [HZ]. Corresponding to $D = 113$, the surface $S(D)$ is a surface of general

type with geometric genus equal to 4. It is known [VG][page 203] that the canonical map from $S(D)$ is a degree two morphism onto a cubic in \mathbb{P}^3 . So we have a natural involution i acting on $S(D)$, we want to prove that i_* at the level of Chow group of zero cycles is -1 . For our convenience let us denote the surface $S(D)$ by S .

2.1. The Bloch-Kas-Liebarman technique. To prove that i_* acts as -1 on the Chow group of zero cycles of S , we follow the Bloch-Kas-Lieberman technique as present in [BKL] the proof of Bloch's conjecture for the surfaces with $p_g = 0$ and not of general type. First consider a pencil of elliptic curves on the cubic surface \mathcal{C} , that is a regular morphism from $\mathcal{C} \rightarrow D$, where D is a line in $(\mathbb{P}^3)^\vee$. Let f denote the morphism from S to \mathcal{C} . Then consider the generic fiber of the pencil $\mathcal{C} \rightarrow D$ and further pull it back to S via f . So let η be the generic point of D and let E_η be the geometric generic fiber of $\mathcal{C} \rightarrow D$. Let us denote $f^{-1}(E_\eta)$ by C_η . So we have a morphism from C_η to E_η . Spreading out this morphism we get a dominant morphism from S to E , where E is the elliptic surface over D' , where D' is a smooth projective curve mapping finitely to D . Consider the Jacobian fibration $J \rightarrow D$ corresponding to $E \rightarrow D$. Now fix a multisection Y of S and let π be the morphism from $S \rightarrow D$. Let us have

$$Y \cap \pi^{-1}(t) = \sum_{i=1}^n p_i(t)$$

for t in D . We map C_η to J_η by

$$q \mapsto (q, \dots, q) \mapsto n f_\eta(q) - \sum_{i=1}^n p_i(\pi(q))$$

then spreading this map we have a dominant morphism g from S to J .

Since by the same process we have a dominant map from \mathcal{C} to J and $p_g(\mathcal{C}) = q = 0$, we get that $p_g(J) = q = 0$.

Proposition 2.2. *Let $T(J)$ denote the albanese kernel for J . If $T(J) = 0$, then the involution i acts as -1 on the albanese kernel $T(S)$ for S .*

Proof. Now want to understand the quasi-inverse of g . Let α belong to J that lies over $t \in D'$. So there is a unique point in $q_i(t)$ on E such that $q_i(t) - p_i(t)$ is rationally equivalent to α . Now $g^{-1}(q_i(t)) = \{q'_i(t), q''_i(t)\}$. So we can define λ to be

$$\alpha \mapsto \sum_i (q'_i(t) + q''_i(t)).$$

Now we check that

$$g_*\lambda(\alpha) = g_*(\sum_i q'_i(t) + q''_i(t))$$

which is

$$\begin{aligned} &= \sum_i g_*(q'_i(t)) + g_*(q''_i(t)) \\ &= \sum_i 4n(q_i(t) - p_i(t)) = 4n^2\alpha. \end{aligned}$$

Let q be a point supported on $C_t = \pi^{-1}(t)$ where $\pi : S \rightarrow D$. Then we prove that $2n^2(q + iq)$ is rationally equivalent to zero on C_t . For that let us compute $\lambda g_*(q + iq)$. So we have by definition of

$$g_*(q + iq) = n f_{t*}(q + iq) - 4 \sum_{i=1}^n f_t(p_i(t)),$$

let $f_t(q) = q'$. Then

$$\lambda g_*(q + iq) = \lambda(4nq') - \lambda(4 \sum_{i=1}^n f_t(p_i(t)))$$

which can be re-written as

$$4 \sum_{i=1}^n \lambda(q' - f_t(p_i(t))).$$

Now $\lambda(q' - f_t(p_i(t))) = n(q + iq)$ so putting that in the above we get that

$$4n^2(q + iq) = \lambda g_*(q).$$

Therefore for any zero cycle z of degree zero on S , we have

$$4n^2(z + iz) - \lambda g_*(z) = 0.$$

Suppose z vanishes on $Alb(S)$, then we have $g_*(z)$ vanishing on $Alb(J)$. If $T(J)$ is zero then we can conclude that $g_*(z)$ is rationally equivalent to 0 on J . Composing with λ we get that $4n^2(z + iz) = 0$. By Roitman's theorem [RO][theorem 3.1] it will follow that $z + iz = 0$, so we have $iz = -z$. \square

Now we repeat the proof that $T(J) = 0$ by showing that J is rational following [BKL].

Proposition 2.3.

$$T(J) = 0$$

Proof. We have $p : J \rightarrow D'$ and a section $\sigma : D' \rightarrow J$ by the very definition of a Jacobian fibration. First we observe that $D = \mathbb{P}^1$ as $q(J) = 0$ and $Alb(J)$ maps onto $Alb(D')$. Then by [Sha][VII, section 3] we have

$$K_J = \pi^*(K_{J,\sigma}(D'))$$

also by the adjunction formula we have

$$K_{J,\sigma}(D') = K_{D'} - N$$

where N is the normal bundle to D' in J . Thus we have

$$h^0(J, rK_J) = h^0(J, \pi^*(r(K_{D'} - N))) = h^0(D', r(K_{D'} - N))$$

since $p_g = 0$ we have that

$$h^0(D', K_{D'} - N) = 0.$$

Since D' is \mathbb{P}^1 , the degree of $K_{D'} - N$ is negative. Therefore by Riemann-Roch we have $h^0(D', r(K_{D'} - N)) = 0$ for $r > 0$. Therefore $P_r(J) = 0$ for all $r \geq 1$, in particular $P_2 = 0$, so J is rational. Therefore $T(J) = 0$. \square

2.4. Generalisation of the above technique. The above technique tells us that if we have a $2 : 1$ morphism from a surface S to a rational surface F , on which we have a pencil of elliptic curves then we can apply the above argument due to [BKL]. Let $f : S \rightarrow F$ be a $2 : 1$ morphism and let $E_{\bar{\eta}}$ be the geometric generic fiber of the elliptic pencil on F . Then pre-composing by f we have a pencil of hyperelliptic curves on S . Let $C_{\bar{\eta}}$ be the geometric generic fiber of this pencil. Then we have $f : C_{\bar{\eta}} \rightarrow E_{\bar{\eta}}$. Now consider the spread of $E_{\bar{\eta}}$ to an elliptic fibration E over a smooth projective curve D mapping finitely onto \mathbb{P}^1 . Using f and a multisection Y of S , we can cook up a morphism from $C_{\bar{\eta}}$ to $J_{\bar{\eta}}$, which is dominant, so spreading this morphism we get a dominant morphism g from S to J , where J is the Jacobian fibration associated to E . Also similarly we have a dominant morphism from F to J , which gives us that the geometric genus of J is zero.

Now the above technique gives us a correspondence on $S \times J$, which induces a push-forward at the level of Chow groups, denote it by λ . Then following the above computation we can prove that λg_* is equal to $4n^2(z + iz)$, for z a degree zero cycle in $CH_0(S)$. Now if z belongs to $T(S)$, then z vanishes with respect to the albanese map for S , hence $g_*(z)$ vanishes with respect to the albanese map for J . If $T(J) = 0$ then $g_*(z)$ is rationally

equivalent to zero on J . So composing by λ we get that $\lambda g_*(z)$ is rationally equivalent to 0 on S . Therefore $4n^2(z + iz)$ is rationally equivalent to zero on S . By Roitman's theorem [RO][theorem 3.1] it will follow that $z + iz$ is rationally equivalent to zero, hence $iz = -z$. Finally we finish the proof by showing $T(J)$ is zero following the argument of [BKL].

3. CURVES ON A SURFACE AND MONODROMY

Let F be a smooth, projective, surface over \mathbb{C} . Let us fix an embedding of F inside \mathbb{P}^N . Let τ be an involution acting on F . Let t be a closed point in \mathbb{P}^{N*} , consider the corresponding hyperplane H_t inside \mathbb{P}^N and consider its intersection with F , then we get a curve C_t inside F and $\tau(C_t)$ inside F . By Bertini's theorem, a general such hyperplane section of F will be smooth and irreducible. Now consider two curves $C_t, \tau(C_t)$ in F . Let g be the genus of C_t . Then we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Sym}^g C_t \times \mathrm{Sym}^g(\tau(C_t)) & \longrightarrow & \mathrm{Sym}^{2g} F \\ \downarrow & & \downarrow \\ A_0(C_t) \times A_0(\tau(C_t)) & \longrightarrow & A_0(F) \end{array}$$

Here the morphism from $\mathrm{Sym}^g C_t \times \mathrm{Sym}^g \tau(C_t)$ to $\mathrm{Sym}^{2g} F$ is given by

$$(\sum_i P_i, \sum_j Q_j) = \sum_i P_i + \sum_j Q_j$$

and the homomorphism from $A_0(C_t) \times A_0(\tau(C_t))$ to $A_0(F)$ is given by

$$j_{t, \tau(t)*} = j_{t*} + j_{\tau(t)*}.$$

It is easy to see that the above diagram is commutative (since \mathbb{C} is algebraically closed). By the Abel-Jacobi theorem $A_0(C_t) \times A_0(\tau(C_t))$ is isomorphic to $J(C_t) \times J(\tau(C_t))$. Following the argument of [BG], proposition 6 we get that the kernel of $j_{t, \tau(t)*}$ is a countable union of translates of an abelian subvariety of $J(C_t) \times J(\tau(C_t))$. Call this abelian subvariety A_t . Now consider a Lefschetz pencils D_1 on F (This we can do, since we have fixed an embedding of F in \mathbb{P}^N). Then for a general t on D_1 , the curves $C_t, \tau(C_t)$ will be smooth and irreducible. Now we prove that for a general t , A_t will either be $\{0\}$ or $J(C_t)$ or $J(\tau(C_t))$ or all of $J(C_t) \times J(\tau(C_t))$. Suppose also that $H^3(F, \mathbb{Q}) = 0$.

Theorem 3.1. *For a very general t in D_1 , the abelian variety A_t is either $\{0\}$ or $J(C_t) \times \{0\}$ or $\{0\} \times J(\tau(C_t))$ or the diagonal $J(C_t)$, (induced by the diagonal embedding $C \rightarrow C \times \tau(C)$) or $J(C_t) \times J(\tau(C_t))$.*

Proof. The argument comes from monodromy. We have a natural monodromy representation of the fundamental group of $D_1 \setminus \{0_1, \dots, 0_m\}$ on the Gysin kernels $H^1(C_t, \mathbb{Q})$ and $H^1(\tau(C_t), \mathbb{Q})$ respectively, for a very general t such that $C_t, \tau(C_t)$ are smooth. By theorem 3.27 in [Vo] we have that these monodromy representations are irreducible. So it will follow that the induced representation of $G = \pi_1(D_1 \setminus \{0_1, \dots, 0_m\}, t)$ on $H^1(C_t, \mathbb{Q}) \oplus H^1(\tau(C_t), \mathbb{Q})$ has the following property. Any G invariant subspace of it is either $\{0\}$ or $H^1(C_t, \mathbb{Q})$ or $H^1(\tau(C_t), \mathbb{Q})$ or all of $H^1(C_t, \mathbb{Q}) \oplus H^1(\tau(C_t), \mathbb{Q})$. Consequently, by using the correspondence between Hodge structures of weight one and abelian varieties we have that the only non-trivial proper abelian subvarieties of $J(C_t) \times J(\tau(C_t))$ are either $J(C_t) \times \{0\}$ or $\{0\} \times J(\tau(C_t))$ or the diagonal $J(C_t)$. Now to prove that A_t is either one of these four possibilities we have to show that the Hodge structure corresponding to A_t is G equivariant. So for a very general t we have an abelian subvariety A_t of $J(C_t) \times J(\tau(C_t))$. Now consider the isomorphism of \mathbb{C} with $\overline{\mathbb{C}(t)}$ and view A_t and $J(C_t) \times J(\tau(C_t))$ as abelian varieties over $\overline{\mathbb{C}(t)}$. Let L be the minimal field of definition of A_t and $J(C_t) \times J(\tau(C_t))$ in $\overline{\mathbb{C}(t)}$. Since L is finitely generated over $\mathbb{C}(t)$ and contained in $\overline{\mathbb{C}(t)}$ we have L finite extension of $\mathbb{C}(t)$. Let D' be a curve such that $\mathbb{C}(D')$ is isomorphic to L , respectively and D' maps finitely onto D_1 . Then we have A_t and $J(C_t) \times J(\tau(C_t))$ defined over L and we can spread A_t and $J(C_t) \times J(\tau(C_t))$ over some Zariski open U in D' . Call these spreads as \mathcal{A}, \mathcal{J} . Then throwing out some more points from U we will get that the morphism from \mathcal{A}, \mathcal{J} to U is a proper, submersion of smooth manifolds, if we view everything over \mathbb{C} (again here we use the non-canonical isomorphism $\overline{\mathbb{C}(t)} = \mathbb{C}$). Then by Ehressmann's theorem we have two fibrations $\mathcal{A} \rightarrow U$ and $\mathcal{J} \rightarrow U$. Since any fibration gives rise to a local system and hence a monodromy representation of the fundamental group of $\pi_1(U, t')$ on $H^{2d-1}(A_t, \mathbb{Q}), H^{4g-1}(J(C_t) \times J(\tau(C_t)), \mathbb{Q}) \cong H^1(C_t, \mathbb{Q}) \oplus H^1(\tau(C_t), \mathbb{Q})$ where $d, 2g$ are dimensions of $A_t, J(C_t) \times J(\tau(C_t))$ (here we might have to replace t by t' , but for very two general points the fibers $A_t, A_{t'}$ will be isomorphic, so are $J(C_t) \times J(\tau(C_t))$ and $J(C_{t'}) \times J(\tau(C_{t'}))$). Now $\pi_1(U, t')$ is a finite index subgroup of G . We prove that $H = H^{2d-1}(A_t, \mathbb{Q})$ is a G -equivariant subspace of $H^1(C_t, \mathbb{Q}) \oplus$

$H^1(\tau(C_t), \mathbb{Q})$ (since A_t is a sub-abelian variety of $J(C_t) \times J(\tau(C_t))$, $H^{2d-1}(A_t, \mathbb{Q})$ is a subspace of $H^1(C_t, \mathbb{Q}) \oplus H^1(\tau(C_t), \mathbb{Q})$). Now G acts on $H^1(C_t, \mathbb{Q}) \oplus H^1(\tau(C_t), \mathbb{Q})$ by the Picard-Lefschetz formula, that is

$$\gamma.(\alpha + \beta) = \gamma.\alpha + \gamma.\beta$$

which is equal to

$$\alpha - \langle \alpha, \delta_\gamma \rangle \delta_\gamma + \beta - \langle \beta, \delta_\gamma \rangle \delta_\gamma.$$

Now suppose that $\alpha + \beta$ belongs to H . We have to prove that for all γ in G , $\gamma.(\alpha + \beta)$ belongs to H . Consider

$$(\gamma)^m(\alpha + \beta) = m\alpha - m\langle \alpha, \delta_\gamma \rangle \delta_\gamma + m\beta - m\langle \beta, \delta_\gamma \rangle \delta_\gamma$$

δ_γ is the vanishing cycles corresponding to γ . Since γ^m is in $\pi_1(U, t')$ we have

$$\gamma^m(\alpha + \beta) - m\alpha - m\beta$$

is in H . That would mean that

$$m\langle \alpha, \delta_\gamma \rangle \delta_\gamma + m\langle \beta, \delta_\gamma \rangle \delta_\gamma$$

is in H , by applying the Picard Lefschetz once again we get that

$$\gamma.(\alpha + \beta)$$

is in H . So H is G equivariant, hence it is either $\{0\}$ or $H^1(C_t, \mathbb{Q})$, $H^1(\tau(C_t), \mathbb{Q})$ or all of $H^1(C_t, \mathbb{Q}) \oplus H^1(\tau(C_t), \mathbb{Q})$. So the corresponding A_t will either be zero or $J(C_t) \times \{0\}$ or $\{0\} \times J(\tau(C_t))$ or $J(C_t)$ or $J(C_t) \times J(\tau(C_t))$. \square

This proves that if for one very general t , A_t is one of the above mentioned possibilities then for another very general t' , $A_{t'}$ will achieve the same possibility.

3.2. Countability of the Bi-Gysin kernel.

Theorem 3.3. *Suppose $A_0(F)$ is not isomorphic to the Albanese variety $\text{Alb}(F)$. Consider a Lefschetz pencil on F as before. Then for a very general t , A_t is actually $\{0\}$ or the diagonal $J(C_t)$.*

Proof. Here we want to prove that if $A_0(F)$ is not weakly representable that is it is not isomorphic to the Albanese variety of F then the kernel of the bi-Gysin homomorphism is countable for a very general hyperplane section C_t of F inside \mathbb{P}^N .

For that consider the family $\mathcal{C} \cup \tau(\mathcal{C})$, that is given by

$$\{(x, C_t) : x \in C_t\} \cup \{(x, \tau(C_t)) : x \in \tau(C_t)\}$$

. Then $\mathcal{C}_t \cup \tau(\mathcal{C})_t$ is nothing but $C_t \cup \tau(C_t)$. Suppose that T is the set of all t in \mathbb{P}^{N^\vee} such that $C_t \cup \tau(C_t)$ is smooth. Let us choose a Lefschetz pencil D in \mathbb{P}^{N^\vee} such that a very general member C_t of this Lefschetz pencil has the property that $A^2(C_t) \oplus A^2(\tau(C_t))$ is weakly representable (this is trivially true since C_t is a curve). Then for the geometric generic fiber $C_{\bar{\eta}}$ of D , we have

$$A^2(C_{\bar{\eta}}) \oplus A^2(\tau(C_{\bar{\eta}}))$$

is weakly representable. So there exists a curve $\Gamma_{\bar{\eta}}$ and a correspondence $Z_{\bar{\eta}}$ on $\Gamma_{\bar{\eta}} \times (C_{\bar{\eta}} \cup \tau(C_{\bar{\eta}}))$ such that

$$Z_{\bar{\eta}*} : A^1(\Gamma_{\bar{\eta}}) \rightarrow A^2(C_{\bar{\eta}}) \oplus A^2(\tau(C_{\bar{\eta}}))$$

is surjective. Let $\Gamma_{\bar{\eta}}$, $Z_{\bar{\eta}}$ are defined over some finite extension L of $\mathbb{C}(t)$. Let D' be a smooth projective curve mapping finitely onto D and having function field L . Then we can spread $Z_{\bar{\eta}}$ and $\Gamma_{\bar{\eta}}$ over some Zariski open U' in D' to get \mathcal{Z}, \mathcal{G} , such that we have

$$\mathcal{Z}_* : A^1(\mathcal{G}) \rightarrow A^2(\mathcal{C}_{U'} \cup \tau(\mathcal{C})_{U'}) .$$

Compactifying and resolving singularities we get \mathcal{Z}'_* and \mathcal{G}' such that

$$\mathcal{Z}'_* : A^1(\mathcal{G}') \rightarrow A^2(\mathcal{C}_{D'} \cup \tau(\mathcal{C})_{D'}) .$$

We have the following commutative diagram.

$$\begin{array}{ccc} A^1(\mathcal{G}') & \longrightarrow & A^2(\mathcal{C}'_{D'} \cup \tau(\mathcal{C})_{D'}) \\ \downarrow & & \downarrow \\ A^1(\Gamma_{\bar{\eta}}) & \longrightarrow & A^2(\mathcal{C}_{\bar{\eta}} \cup \tau(\mathcal{C}_{\bar{\eta}})) \end{array}$$

By a diagram chase [Ba][page 121-124] we can prove that for any α in $A^2(\mathcal{C}_{D'} \cup \tau(\mathcal{C})'_{D'})$ there exists n_α an integer such that $n_\alpha \alpha$ belongs to the subgroup generated by the image of \mathcal{Z}'_* and the kernel of the pull-back $A^2(\mathcal{C}_{D'} \cup \tau(\mathcal{C})_{D'})$ to $A^2(\mathcal{C}_{\bar{\eta}} \cup \tau(\mathcal{C}_{\bar{\eta}}))$. This is by theorem 4.7.1 (page 126) in [Ba]. Now the kernel of the pull-back $A^2(\mathcal{C}_{D'} \cup \tau(\mathcal{C})_{D'}) \rightarrow A^2(\mathcal{C}_{\bar{\eta}} \cup \tau(\mathcal{C}_{\bar{\eta}}))$ tensored with \mathbb{Q} is the same as the kernel $A^2(\mathcal{C}_{D'} \cup \tau(\mathcal{C})_{D'}) \otimes \mathbb{Q} \rightarrow A^2(\mathcal{C}_{\bar{\eta}} \cup \tau(\mathcal{C}_{\bar{\eta}})) \otimes \mathbb{Q}$. The later group in the above is the colimit of $A^2(\mathcal{C}_{W'} \cup \tau(\mathcal{C})_{W'}) \otimes \mathbb{Q}$, W' Zariski open in D' . So by the localization exact sequence the kernel is the direct sum of images of the bi-Gysin homomorphisms

$$(j_{t*} + j_{\tau(t)*}) : \oplus_{t' \in D'} (A^1(C_{t'}) \oplus A^1(\tau(C_{t'}))) \otimes \mathbb{Q} \rightarrow A^2(\mathcal{C}_{D'} \cup \tau(\mathcal{C})_{D'}) \otimes \mathbb{Q} .$$

Arguing as in [Ba][Theorem 4.7.1] we can prove that if the bi-Gysin homomorphism is zero for a very general t , then actually it is zero for a general t . Also observe that if $j_{t*} = 0$ then $j_{\tau(t)*} = 0$. So supposing that the bi-Gysin homomorphism is zero or one of j_{t*} or $j_{\tau(t)*}$ is zero we get that from the above

$$A^2(\mathcal{C}_{D'} \cup \tau(\mathcal{C})_{D'}) \otimes \mathbb{Q}$$

is weakly representable. That will imply that $A^2(F) \otimes \mathbb{Q}$ is weakly representable, hence so is $A^2(F)$. That will be a contradiction to our assumption. So if $A^2(F)$ is not weakly representable then for a very general t , the kernel of the bi-Gysin homomorphism is countable. \square

3.4. Surfaces with $p_g > 0, q = 0$. Similar techniques as above will show that the kernel of $j_{t*} - j_{\tau(t)*}$ is either $\{0\}$ or the diagonal $J(C_t)$ for a very general t , if $A^2(F)$ is not isomorphic to the albanese variety of F , which is the case for surfaces with $p_g > 0, q = 0$. Note that both the kernels $j_{t*} + j_{\tau(t)*}$ and $j_{t*} - j_{\tau(t)*}$ cannot be the diagonal $J(C_t)$. If it is the case then for any z in $J(C_t)$, we have

$$j_{t*}(z) = \tau(j_{t*})(z) = -j_{t*}(z)$$

which implies that

$$2j_{t*}(z) = 0$$

since $q = 0$ and Roitman's theorem tells us $j_{t*}(z) = 0$, so j_{t*} is the zero map for a very general t . This contradicts the fact that the kernel of j_{t*} is countable for a very general t , provided that $A^2(F)$ is not isomorphic to the albanese variety of F , [BG][Theorem 19]. Now since $A^2(F)$ is generated by cycles supported on $J(C_t)$, where C_t is a very general smooth hyperplane section of F . So either for a very general C_t , kernel of both $j_{t*} + j_{\tau(t)*}, j_{t*} - j_{\tau(t)*}$ are countable or one of the kernel is $J(C_t)$. So either both the kernels are countable or τ_* acts as id or $-id$ on the image of $J(C_t)$ inside $A^2(F)$. If the later possibility happens then the action of τ_* on $A^2(F)$, can be detected from its action on the image of the Jacobian of the very general hyperplane section C_t . For that we have to exclude the possibility that the kernels mentioned above, cannot both be countable.

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